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A coordinate system for the Teichmüller space of a compact surface and a rational represesentation of the mapping class group

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§1. Teichmüller spaces and mapping class groups. Let $S=S_{g,n}$ denote a compact oriented surface of genus g with n boundary curves $c_1,\ldots, c_n$. We assume that $2g-2+n>0$. The fundamental group $\Gamma_{g,n}=\pi_1(S)$ has the presentation:

$$\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n : (\prod_{j=1}^{g}[a_j, b_j])c_1\cdots c_n=1\rangle,$$

where $[a,b]=aba^{-1}b^{-1}$ is the commutator of a and b, and we denote also by c_j the homotopy class of c_j. Let $L=(L_1,\ldots,L_n)\in\mathbb{R}^n_{\geq 0}$ and $\mathbb{T}_{g,n}(L)$ be the Teichmüller space of isotopy classes of complete hyperbolic metrics on the interior $I(S)$ of S with the length of the geodesic isotopic to c_j is L_j for $j=1,\ldots,n$ (c_j corresponds to a puncture if $L_j=0$.) Let $\mathcal{C}=\mathcal{C}_{g,n}$ denote the set of isotopy classes of unoriented closed curves in $I(S)$. Each $\gamma\in\mathcal{C}$ defines a real analytic function on $\mathbb{T}_{g,n}(L)$ called the *geodesic length function* associated to γ: For each $X\in\mathbb{T}_{g,n}(L)$

$$\ell_\gamma(X)=\text{the length of the geodesic representation in } \gamma \text{ on } X.$$

We also define $\tau_\gamma(X)=2\cosh(\ell_\gamma(X)/2)$. X defines a Fuchsian representation χ of $\Gamma_{g,n}$ into $PSL(2,\mathbb{R})$ up to conjugacy and we have

$$\tau_\gamma(X)=|\mathrm{tr}\chi(\gamma)|.$$

We call τ_γ the *trace function* associated to γ. We can identify $X\in\mathbb{T}_{g,n}(L)$ with the simultaneous conjugacy class $\mathcal{G}(X)$ of a tuple of matrices in $SL(2,\mathbb{R})$

$$(A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_n)=(\chi(a_1), \chi(b_1), \ldots, \chi(a_g), \chi(b_g), \chi(c_1), \ldots, \chi(c_n))$$

with $\mathrm{tr}A_j>0$, $\mathrm{tr}B_j>0$ ($j=1,\ldots,g$) and $\mathrm{tr}C_j=-2\cosh(L_j/2)=-\ell_j<0$ ($j=1,\ldots,n$), and hence identify $\mathbb{T}_{g,n}(L)$ with

$$\mathcal{T}_{g,n}(\ell_1,\ldots,\ell_n)=\{\mathcal{G}(X):X\in\mathbb{T}_{g,n}(L)\}.$$

The Teichmüller space $\mathbb{T}_{g,n}(L)$ is homeomorphic to $\mathbb{R}^d$, where $d = 6g - 6 + 2n$.

Let $\mathcal{MC}_{g,n}$ denote the *mapping class group* of the surface $S = S_{g,n}$. Each element $[f]$ of $\mathcal{MC}_{g,n}$ is the isotopy class of an orientation preserving diffeomorphism $f : S \to S$ preserving each boundary curve setwise. $\mathcal{MC}_{g,n}$ acts on the Teiichmüller space $\mathbb{T}_{g,n}(L)$. If $X = (S, \sigma) \in \mathbb{T}_{g,n}(L)$, where σ is a hyperbolic metric on S, then $[f](X)$ is the isotopy class of $(S, f^*\sigma)$. This group induces a subgroup of outer automorphisms of the surface group $\Gamma_{g,n}$.

The first statement of the following theorem is proved by Schmutz, Okumura, Feng Luo and others. For a proof of the full statement, see [8].

Theorem 1 *There are simple closed curves $\gamma_1,..., \gamma_{d+1}$ on $I(S)$ such that*

$$\Phi : \mathbb{T}_{g,n}(L) \to \mathbb{R}^{d+1}$$

defined by $\Phi(X) = (\tau_{\gamma_1}(X), ..., \tau_{\gamma_{d+1}}(X))$ is an embedding. Moreover, the mapping class group $\mathcal{MC}_{g,n}$ acts on $\Phi(\mathbb{T}_{g,n}(L))$ as a group of rational transformations in the coordinates $x_1,...., x_{d+1}$ of $\mathbb{R}^{d+1}$ and $\ell_1,...,\ell_n$ over the rational number field.

§2. Finite subgroups of the mapping class group of genus 2 surface. For the rest of this note, $\mathbb{T}_g$ means the Teichmüller space of the closed surface of genus g. By the Nielsen-Kerckhoff realization theorem [5], each finite subgroup G of $\mathcal{MCG}_g = \mathcal{MCG}_{g,0}$ acts on a Riemann surface R of genus g as a group of conformal automorphisms. For each $\varphi \in \mathcal{MCG}_g$, let φ_* denote the rational transformation acting on $\Phi(\mathbb{T}_g)$ obtained by Theorem 1. Let $x_0 = \Phi(X_0)$ be an arbitrary point of $\Phi(\mathbb{T}_g)$. If $\varphi_*^m(x_0) = x_0$ for some $m > 0$, then φ is an isotopy class of a conformal automorphism (including the identity map) on the Riemann surface X_0 and we can conclude that φ is *elliptic* or it has a finite order. Since the order of an elliptic element is at most a number Pg depending only on g ($\leq 84(g-1)$ by Riemann-Hurwitz formula), we can detect whether an element of $\mathcal{MC}_g$ is ellptic or not by showing some φ_*^m $(1 \leq m \leq P_g)$ fixes x_0.

Let G be a finite subgroup of $\mathcal{MC}_g$ and assume that all elements of G fix a Riemann surface R of genus g. If the genus of the factor surface R/G is h and the covering map $\pi : R \to R/G$ is branched over n points $p_1,..., p_n$ with branching orders m_j with $m_1 \leq m_2 \leq \cdots \leq m_n$, then $(h; m_1, ..., m_n)$ is the *type* of the orbifold R/G. In stead of $(h, m_1, ..., m_n)$, we often write $(h; \nu_1^{r_1}, ..., \nu_p^{r_p})$ $(\nu_1 < \cdots < \nu_p)$ if ν_j appears r_j times in $(m_1, ..., m_n)$.

The mapping class group $\mathcal{MCG}_2$ of a closed orientable surface of genus 2 is generated by Dehn twists ω_1, ω_2, ω_3, ω_4 and ω_5 with the following defining relations (see [1, p.184]):

$$\omega_i\omega_j = \omega_j\omega_i \quad \text{if } |i-j| \geq 2,\ 1 \leq i,j \leq 5 \tag{1}$$
$$\omega_i\omega_{i+1}\omega_i = \omega_{i+1}\omega_i\omega_{i+1} \qquad (1 \leq i \leq 4) \tag{2}$$
$$(\omega_1\omega_2\omega_3\omega_4\omega_5)^6 = 1 \tag{3}$$
$$(\omega_1\omega_2\omega_3\omega_4\omega_5^2\omega_4\omega_3\omega_2\omega_1)^2 = 1 \tag{4}$$
$$\omega_1\omega_2\omega_3\omega_4\omega_5^2\omega_4\omega_3\omega_2\omega_1 \text{ and } \omega_i \text{ commute for } i = 1,2,3,4,5 \tag{5}$$

In [2] S. A. Broughton classified completely the finite subgroups of $\mathcal{MCG}_2$, up to topological equivalence. After a lengthy calculations, Nakamura and the author found explicit expressions by the Dehn twists ω_1, ..., ω_5 for the generator-systems in Broughton's list.

Theorem 2 ([9]). *A non-trivial finite subgroup of $\mathcal{MCG}_2$ of a closed orientable surface of genus 2 is conjugate with one of the groups in the table below.*

The table shows the group G_* corresponding to (2,$*$) in [2] with generators expressed in ω_1, ... , ω_5, the order $|G_*|$ and the orbifold type.

(2.a) $G_a = \langle x : x^2 = 1\rangle \cong \mathbb{Z}_2$, $x = \omega_1\omega_2\omega_3\omega_4\omega_5^2\omega_4\omega_3\omega_2\omega_1$, 2, $(0;2^6)$.

(2.b) $G_b = \langle x : x^2 = 1\rangle \cong \mathbb{Z}_2$, $x = (\omega_1\omega_2\omega_3\omega_4\omega_5)^3$, 2, $(1;2^2)$.

(2.c) $G_c = \langle x : x^3 = 1\rangle \cong \mathbb{Z}_3$, $x = (\omega_1\omega_2\omega_3\omega_4\omega_5)^2$, 3, $(0;3^4)$.

(2.e) $G_e = \langle x : x^4 = 1\rangle \cong \mathbb{Z}_4$, $x = (\omega_1\omega_1\omega_2\omega_3\omega_4)^2$, 4, $(0;2^2,4^2)$.

(2.f) $G_f = \langle x : x^2 = y^2 = [x,y] = 1\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $x = \omega_1\omega_2\omega_3\omega_4\omega_5^2\omega_4\omega_3\omega_2\omega_1$. $y = (\omega_1\omega_2\omega_3\omega_4\omega_5)^3$, 4, $(0;2^5)$.

(2.h) $G_h = \langle x : x^5 = 1\rangle \cong \mathbb{Z}_5$, $x = (\omega_1\omega_2\omega_3\omega_4)^2$, 5, $(0;5^3)$.

(2.i) $Gi = \langle x : x^6 = 1\rangle \cong \mathbb{Z}_6$, $x = \omega_1\omega_2\omega_3\omega_4\omega_5$, 6, $(0,3,6^2)$.

(2.k.1) $G_{k1} = \langle x : x^6 = 1\rangle \cong \mathbb{Z}_6$, $x = \omega_1\omega_2\omega_5^{-1}\omega_4^{-1}$, 6, $(0,2^2,3^2)$.

(2.k.2) $G_{k2} = \langle x, y : x^2 = y^3 = 1, xyx^{-1} = y^{-1} \rangle \cong D_3$, $x = (\omega_1\omega_2\omega_3\omega_4\omega_5)^3$, $y = (\omega_1\omega_2\omega_5^{-1}\omega_4^{-1})^2$, 6, $(0, 2^2, 3^2)$.

(2.l) $G_l = \langle x : x^8 = 1 \rangle \cong \mathbb{Z}_8$, $x = \omega_1\omega_1\omega_2\omega_3\omega_4$, 8, $(0; 2, 8, 8)$.

(2.m) $G_m = \langle x, y : x^4 = y^4 = 1, x^2 = y^2, xyx^{-1} = y^{-1} \rangle \cong \tilde{D}_2$, $x = (\omega_1\omega_2\omega_1\omega_3\omega_4)^2$, $y = (\omega_2\omega_3\omega_5\omega_4\omega_3)^2$, 8, $(0; 4, 4, 4)$.

(2.n) $G_n = \langle x, y : x^2 = y^4 = 1, xyx^{-1} = y^{-1} \rangle \cong D_4$, $x = (\omega_1\omega_2\omega_3\omega_4\omega_5)^3$, $y = (\omega_1\omega_2\omega_4\omega_3\omega_2)^2$, 8, $(0, 2^3, 4)$.

(2.o) $G_o = \langle x : x^{10} = 1 \rangle \cong \mathbb{Z}_{10}$, $x = \omega_1\omega_2\omega_3\omega_4$, 10, $(0, 2, 5, 10)$.

(2.p) $G_p = \langle x, y : x^2 = y^6 = [x, y] = 1 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_6$, $x = \omega_1\omega_2\omega_3\omega_4\omega_5^2\omega_4\omega_3\omega_2\omega_1$, $y = \omega_1\omega_2\omega_3\omega_4\omega_5$, 12, $(2, 6, 6)$.

(2.r) $G_r = \langle x : x^4 = y^3 = 1, xyx^{-1} = y^{-1} \rangle \cong D_{4,3,-1}$, $x = (\omega_1\omega_2\omega_4\omega_3\omega_2)^2$, $y = (\omega_1\omega_2\omega_3\omega_4\omega_5)^2$, 12, $(0, 3, 4^2)$.

(2.s) $G_s = \langle x, y : x^2 = y^6 = 1, xyx^{-1} = y^{-1} \rangle \cong D_6$, $x = (\omega_1\omega_2\omega_3\omega_4\omega_5)^3$, $y = \omega_1\omega_2\omega_5^{-1}\omega_4^{-1}$, 12, $(0, 2^3, 3)$.

(2.u) $G_u = \langle x, y : x^2 = y^8 = 1, xyx^{-1} = y^3 \rangle \cong D_{2,8,3}$, $x = (\omega_1\omega_2\omega_3\omega_4\omega_5)^3$, $y = \omega_1\omega_2\omega_4\omega_3\omega_2$, 16, $(0, 2, 4, 8)$.

(2.w) $G_w = \left\langle x, y, z, w : \begin{array}{l} x^2 = y^2 = z^2 = w^3 = [y, z] = [y, w] = [z, w] = 1 \\ xyx^{-1} = y, xzx^{-1} = zy, xwx^{-1} = w^{-1} \end{array} \right\rangle \cong$ $\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$, $x = (\omega_1\omega_2\omega_1\omega_4^{-1}\omega_5^{-1}\omega_4^{-1})(\omega_1\omega_2\omega_3\omega_4\omega_5)^3$, $y = \omega_1\omega_2\omega_3\omega_4\omega_5^2\omega_4\omega_3\omega_2\omega_1$, $z = (\omega_1\omega_2\omega_3\omega_4\omega_5)^3$, $w = (\omega_1\omega_2\omega_3\omega_4\omega_5)^4$, 24, $(0, 2, 4, 6)$.

(2.x) $G_x = \langle x, y : x^3 = y^4 = 1, xy^2 = y^2x, (xy)^3 = 1 \rangle \cong SL_2(3)$, $x = (\omega_2\omega_1\omega_4^{-1}\omega_5^{-1})(\omega_1\omega_2\omega_3\omega_4\omega_5^2\omega_4\omega_3\omega_2\omega_1)$, $y = (\omega_1\omega_2\omega_1\omega_3\omega_4)^2$, 24, $(0, 3^2, 4)$

(2.aa) $G_{xx} = \left\langle x, y, u : \begin{array}{l} x^3 = y^4 = (xy)^3 = 1, xy^2 = y^2x, u^2 = xy^{-1}x^{-1}y^2 \\ uxu^{-1} = y^{-1}x^{-1}y, uyu^{-1} = x^{-1}yx \end{array} \right\rangle \cong$ $GL_2(3)$, $x = (\omega_2\omega_1\omega_4^{-1}\omega_5^{-1}\omega_4^{-1})(\omega_1\omega_2\omega_3\omega_4\omega_5^2\omega_4\omega_3\omega_2\omega_1)$, $y = (\omega_1\omega_2\omega_1\omega_3\omega_4)^2$, $u = \omega_2\omega_3\omega_5\omega_4\omega_3$, 48, $(0, 2, 3, 8)$

For general $g > 1$, the mapping class group $\mathcal{MC}_g$ is generated $2g + 1$ Dehn twists ω_0, ω_1,..., ω_{2g} called *Humphries generators* (See Theorem 4.14

and Figure 4.5 in [3]) such that the same relations as in (1) and (2) hold and $\zeta^{2g+2} = \eta^{4g+2} = 1$, where, with an additional Dehn twist ω_{2g+1} about a curve $c_{2g+1} = m_g$ in Figure 4.5 in [3],

$$\zeta = \omega_1\omega_2\cdots\omega_{2g+1}, \quad \eta = \omega_1\omega_2\cdots\omega_{2g}.$$

We have by (1) and (2)

$$\begin{aligned} \omega_2\zeta &= \omega_1\omega_2\omega_1(\omega_3\cdots\omega_{2g+1}) \\ &= (\omega_1\omega_2\cdots\omega_{2g+1})\omega_1 = \zeta\omega_1 \end{aligned}$$

and likewise

$$\omega_{i+1}\zeta = \zeta\omega_i \quad \text{for } i = 1, ..., 2g. \tag{6}$$

By using this we have also that

$$\begin{aligned} \omega_1\zeta &= \zeta\zeta^{-1}\omega_1\zeta \\ &= \zeta\omega_{2g+1}^{-1}\omega_{2g}^{-1}\cdots\omega_2^{-1}\zeta \\ &= \zeta\omega_{2g+1}^{-1}\omega_{2g}^{-1}\cdots\omega_3^{-1}\zeta\omega_1^{-1} \\ &\vdots \\ &= \zeta^2\omega_{2g}^{-1}\omega_{2g-1}^{-1}\cdots\omega_1^{-1} = \zeta^2\eta^{-1}. \end{aligned}$$

and hence $\omega_1 = \zeta^2\eta^{-1}\zeta^{-1}$. Then by (6)

$$\omega_2 = \zeta^3\eta^{-1}\zeta^{-2},\ \omega_3 = \zeta^4\eta^{-1}\zeta^{-3},\ \dots,\ \omega_{2g+1} = \zeta^{2g+2}\eta^{-1}\zeta^{-2g-1} = \eta^{-1}\zeta^{-2g-1}.$$

If $g = 2$, then $c_0 = c_5$ and hence we obtain Korkmaz's theorem [6] for $g = 2$.

Theorem 3 *The mapping class group $\mathcal{MC}_2$ is generated by $\zeta = \omega_1\omega_2\omega_3\omega_4\omega_5$ and $\eta = \omega_1\omega_2\omega_3\omega_4$ satisfying $\zeta^6 = \eta^{10} = 1$.*

Hirose obtained expressions by Dehn twists of all torsions in the mapping class group $\mathcal{MC}_g$ with $g \leq 4$ in [4].

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